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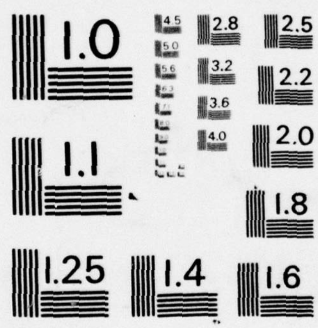

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ON A CLASS OF CONCAVE-SEPARABLE INTEGER PROGRAMS

BY

GARY A. KOCHMAN

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ON A CLASS OF CONCAVE-SEPARABLE INTEGER PROGRAMS

by

Gary Kochman

In this paper we consider the class of concave-separable integer programming problems,

$$(S_f) \qquad \text{maximize} \quad \sum_{j=1}^{n} f_j(x_j) \; ,$$

$$\text{subject to}$$

$$Ax = b$$

$$\ell \leq x \leq u, \quad x \text{ integer} \; ,$$

where A is an $m \times n$ unimodular matrix (see Section 1), b is an m-vector, ℓ, u, and $x = (x_1, \ldots, x_n)^T$ are n-vectors, and each $f_j(x_j)$ is concave for $\ell_j \leq x_j \leq u_j$, $j = 1, 2, \ldots, n$. Without loss of generality, b, ℓ, and u may be taken to be all-integer.

Practical interest in problems of the type (S_f) appears to have originated with the work of Schwartz and Dym (1971). In their model, the $f_j(x_j)$ take the specific form

$$f_j(x_j) = v_j(1-d^{x_j}), \quad j = 1, 2, \ldots, n \; ,$$

where $v_j > 0$ and $0 < d < 1$ are constants. Furthermore, the matrix A consists of a single row with each entry equal to one. Weinstein and Yu (1973) have generalized the Schwartz and Dym model to include problems in which each objective function component $f_j(x_j)$ may be any concave, nonnegative, increasing function. In the present work,

the latter two assumptions are dropped as well, and we further extend the class to include problems with general unimodular constraint matrices.

Other applications which give rise to problems of the form (S_f) have been considered separately by Kochman (1976) and McCallum (1974). Following a brief discussion of unimodularity in Section 1, a general solution procedure for concave-separable integer programs is developed in Section 2. The procedure is illustrated in Section 3 using the models discussed by McCallum and by Schwartz and Dym.

1. Unimodularity

To begin, a few concepts related to unimodular matrices are presented. The notation "det X" is used to denote the determinant of the square matrix X.

Definition: Let the $m \times n$ matrix M have all-integer entries and rank m. Then M is unimodular if each basis B_M of M satisfies $\det B_M = \pm 1$.

Theorem 1: Let the $m \times n$ matrix A be unimodular, and denote the columns of A by a_j, $j = 1, 2, \ldots, n$. Consider the matrix $\tilde{A}$ formed from A by appending columns $a_j^{(k)}$, $k = 1, 2, \ldots, K_j$, where each $a_j^{(k)}$ is identically equal to a_j for all $k = 1, 2, \ldots, K_j$, and K_j is arbitrary but finite. Then $\tilde{A}$ is also unimodular.

Proof: For each basis $B_{\tilde{A}}$ of $\tilde{A}$, there is a corresponding identical basis B_A of A. That is, each of the columns of $B_{\tilde{A}}$ is identical to one of the columns of B_A, although the order in which the columns appear may differ. Thus, $\det B_{\tilde{A}} = \pm \det B_A = \pm 1$. Q.E.D.

The following result due to Veinott and Dantzig (1968) is included for completeness.

<u>Theorem 2</u>: Let the $m \times n$ matrix A have rank m, and let $X(A,b) \equiv \{x | Ax = b, x \geq 0\}$. Suppose A has all-integer entries. Then the following are equivalent:

(i) A is unimodular;

(ii) The extreme points of $X(A,b)$ are integral for all integral b;

(iii) Every basis has an integral inverse.

For our purposes, parts (i) and (ii) of Theorem 2 are of greatest interest. Because of their equivalence, these imply that when the constraint matrix A is unimodular, the linear program,

$$\begin{aligned} &\text{maximize} \quad cx \ , \\ &\text{subject to} \\ &\qquad\qquad Ax = b \\ &\qquad\qquad x \geq 0 \ , \end{aligned}$$

has an optimal solution x^* which is naturally integral.

2. A General Solution Procedure

Returning now to the problem (S_f), consider the piecewise linear approximation to $f_j(x_j)$ which coincides with $f_j(x_j)$ at each integer point between ℓ_j and u_j. (See Figure 1.) Call this approximation $g_j(x_j)$. It is easy to see that $g_j(x_j)$ is concave whenever $f_j(x_j)$ is concave.[1/]

[1/] Actually, for the results here, it is not necessary that $f_j(x_j)$

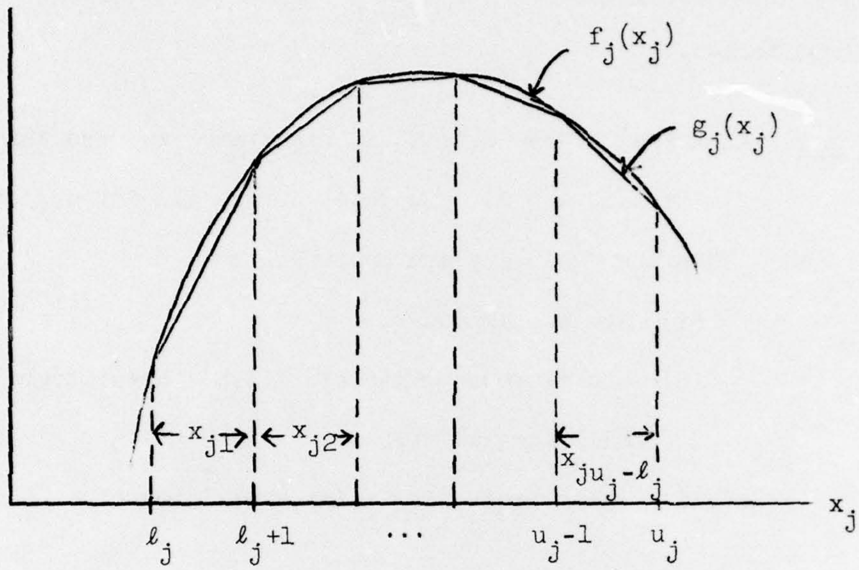

Figure 1: Form of $g_j(x_j)$, the piecewise linear approximation to $f_j(x_j)$

Define problem (S_g) by

$$(S_g) \qquad \text{maximize} \quad \sum_{j=1}^{n} g_j(x_j) \; ,$$

subject to

$$Ax = b$$

$$\ell \leq x \leq u, \quad x \text{ integer} \; ,$$

where A, b, ℓ, and u are as in (S_f). Since only integer solutions to (S_f) are of interest, and since $g_j(x_j)$ differs from $f_j(x_j)$ only at non-integer values of x_j, it is clear that (S_g) and (S_f) must have the same optimal solution(s). Consequently, (S_g) and (S_f) are equivalent integer programs, and (S_g) may be solved in place of (S_f).

be concave. It is sufficient that $f_j(x_j)$ "looks" concave at the integer points in its domain; i.e., that the approximating function $g_j(x_j)$ $(j = 1, 2, \ldots, n)$ is concave.

As indicated in Figure 1, each variable x_j can be expressed as the sum of $(u_j - \ell_j)$ auxiliary variables x_{jk},

$$x_j = \ell_j + \sum_{k=1}^{u_j-\ell_j} x_{jk}, \qquad j = 1, 2, \ldots, n, \tag{1}$$

provided, of course, that $0 \leq x_{jk} \leq 1$ $(k = 1, 2, \ldots, u_j - \ell_j)$, and that no $x_{jk} > 0$ unless $x_{jk-1} = 1$. Each of the auxiliary variables x_{jk}, $k = 1, 2, \ldots, u_j - \ell_j$, is associated with one of the linear segments of the approximating function $g_j(x_j)$, and the integer restrictions on the x_j $(j = 1, 2, \ldots, n)$ are recaptured by requiring each x_{jk} to equal either zero or one. Designating the slope of the k^{th} segment of $g_j(x_j)$ by s_{jk},

$$s_{jk} \equiv \frac{g_j(\ell_j+k) - g_j(\ell_j+k-1)}{(\ell_j+k) - (\ell_j+k-1)} = g_j(\ell_j+k) - g_j(\ell_j+k-1),$$

or

$$s_{jk} = f_j(\ell_j+k) - f_j(\ell_j+k-1), \quad \text{for } k = 1,2,\ldots,u_j-\ell_j,\ j = 1,2,\ldots,n. \tag{2}$$

Using (1), (2), $g_j(x_j)$ may be expressed as

$$g_j(x_j) = g_j(\ell_j) + \sum_{k=1}^{u_j-\ell_j} s_{jk}x_{jk}, \qquad j = 1, 2, \ldots, n. \tag{3}$$

Thus, problem (S_g) may be re-written as

$$(T) \quad \sum_{j=1}^{n} g_j(\ell_j) + \max \sum_{j=1}^{n} \left(\sum_{k=1}^{u_j-\ell_j} s_{jk}x_{jk} \right),$$

subject to

$$\sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{u_j-\ell_j} x_{jk} \right) = b_i', \qquad i = 1, 2, \ldots, m$$

$$0 \leq x_{jk} \leq 1,\ x_{jk} \text{ integer},\ k = 1, 2, \ldots, u_j - \ell_j$$
$$j = 1, 2, \ldots, n,$$

where $b_i' = b_i - \sum_{j=1}^{n} a_{ij}\ell_j$, and the b_i and a_{ij} are the entries of b and A, respectively. The adjusted right-hand sides b_i', $i = 1, 2, \ldots, m$, are integral, since each b_i, a_{ij}, and ℓ_j are integral.

Letting $\tilde{A}$ denote the matrix of constraint coefficients for (T), it is clear that the column of $\tilde{A}$ associated with each x_{jk} $(1 \leq k \leq u_j - \ell_j)$ is identical to the corresponding column of A associated with x_j. That is, the matrix $\tilde{A}$ is relat[illegible] to the matrix A in that for each column a_j of A, there appear $u_j - \ell_j$ identical columns in $\tilde{A}$. It follows from Theorem 1 and the unimodularity of A that $\tilde{A}$ is unimodular. Thus, the integer restrictions on the auxiliary variables x_{jk} may be dropped, and (T) may be solved as a linear program. The unimodularity of $\tilde{A}$ and Theorem 2 assure that the optimal solution, $x_{jk} = x_{jk}^*$ $(1 \leq k \leq u_j - \ell_j,\ 1 \leq j \leq n)$, will be all-integer. The optimal integer solution x^* to (S_g) - and so, to the original problem (S_f) - may then be recovered by (1).

In practice, it is not necessary to introduce the auxiliary variables x_{jk} explicitly. From the theory of concave-separable linear programs, it is known that no x_{jk} will be increased above zero unless $x_{jk-1} = 1$. Thus, using the revised simplex method (generating the columns of $\tilde{A}$ only as needed) with the bounded-variable technique, and setting markers k_j, $j = 1, 2, \ldots, n$, to point to the first x_{jk} at its lower bound of zero, only x_{jk_j} and x_{jk_j-1} are candidates to enter the basis. Hence, only the columns associated with x_{jk_j}, x_{jk_j-1} $(j = 1, 2, \ldots, n)$

need be priced out at each iteration. These columns differ only in the objective row, so there is little additional work in updating both rather than one. Note that the s_{jk} may be computed by (2) only as needed. This can be important computationally if the functions $f_j(x_j)$ are difficult to evaluate. We illustrate these remarks with two simple examples in the next section.

3. Examples

There are several known applications which give rise to concave-separable integer programs, and two of these are now discussed in more detail. The first is a slight generalization of the weapons allocation model of Schwartz and Dym:

$$
\begin{aligned}
&\text{maximize} \quad \sum_{j=1}^{n} v_j(1-d_j^{x_j}) \,, \\
&\text{subject to} \\
&\qquad \sum_{j=1}^{n} x_j = M \\
&\qquad x_j \geq 0, \ \text{integer, for } j = 1, 2, \ldots, n \,.
\end{aligned}
\tag{4}
$$

The problem (4) is that of allocating M weapons to n targets, where

x_j = number of weapons allocated to target j;
v_j = value of destroying j^{th} target $(v_j > 0,\ 1 \leq j \leq n)$; and
d_j = probability that a weapon allocated to the j^{th} target will fail to destroy it $(0 < d_j < 1)$.

Schwartz and Dym developed a simple rounding procedure by which the optimal integer solution to (4) could be obtained from the optimal

continuous solution.[1/] It is assumed that $d_j = d$ for all j, although the authors point out that, in general, the true probabilities d_j would be different for different j. The rounding procedure employed, however, is valid only for the invariant case $d_j = d$ = constant for $1 \leq j \leq n$. Here, this restriction is removed.

It is easy to verify that (4) belongs to the class of problems represented by (S_f). The concavity of $f_j(x_j) \equiv v_j(1-d_j^{x_j})$ follows immediately by noting that

$$\frac{d^2 f_j(x_j)}{dx_j^2} = -v_j d_j^{x_j} (\ell n\ d_j)^2 < 0, \quad \text{for all } x_j .$$

Furthermore, the "matrix" of constraint coefficients is clearly unimodular since each constraint coefficient, and therefore the determinant of each basis, is identically equal to one.

In problem (4), there are no explicit upper bounds on the x_j, and the lower bounds are $\ell_j = 0$, $j = 1, 2, \ldots, n$. Thus, $g_j(\ell_j) = f_j(0) = 0$, for $j = 1, 2, \ldots, n$. The associated linear program (T) in this case is

$$\begin{aligned} &\text{maximize} \quad \sum_{j=1}^{n} \sum_{k=1}^{M} s_{jk} x_{jk} , \\ &\text{subject to} \\ &\qquad \sum_{j=1}^{n} \sum_{k=1}^{M} x_{jk} = M \\ &\qquad 0 \leq x_{jk} \leq 1, \quad \text{for } j = 1, 2, \ldots, n, k = 1, 2, \ldots, M . \end{aligned} \tag{5}$$

1/ McGill (1970) has since shown that the Schwartz and Dym model is a special case of a class of problems for which the optimal integer solution can be obtained by rounding the optimal continuous solution appropriately.

Problem (5) is a bounded-variable linear programming knapsack problem. Briefly, such a problem is solved by computing the ratio of the objective function coefficient to the constraint coefficient for each variable. Each variable is initially set to its lower bound and then, in turn, increased to its upper bound in order of decreasing ratios until the single functional constraint is satisfied. Thus, the last variable to be changed is increased only so far as possible (i.e., not necessarily to its upper bound).[1/]

Since each constraint coefficient in (5) is unity, however, the solution procedure reduces to simply increasing the x_{jk} in order of decreasing s_{jk}. As discussed in Section 2, the concavity of $f_j(\cdot)$ eliminates the need for introducing the auxiliary variables x_{jk} explicitly. In comparing the slopes s_{jk} to determine which x_{jk} (i.e., which x_j) should be increased next, only s_{jk_j} need be considered for each j, where, as before, k_j $(1 \leq j \leq n)$ indexes the first x_{jk} not yet increased to unity. The concavity of $f_j(\cdot)$ ensures that $s_{j\ell} \leq s_{jk_j}$ for all $\ell \geq k_j$.

Using (2), it is easy to show that the s_{jk} in (5) are given by

$$s_{jk} = v_j(1-d_j)d_j^{k-1}, \qquad k = 1, 2, \ldots, M . \tag{6}$$

But from (6), $s_{jk+1} = v_j(1-d_j)d_j^k$, or

$$s_{jk+1} = d_j s_{jk}, \quad \text{for} \quad k = 1, 2, \ldots, M-1 . \tag{7}$$

The algorithm now presented for the Schwartz and Dym model (4)

1/ See Kochman (1976) for a more detailed discussion of the solution of bounded-variable linear programming knapsack problems.

uses (7) rather than (6) to compute the coefficients s_{jk}. It is assumed that $M > 0$, since otherwise the solution is trivial.

Algorithm 1 (Schwartz and Dym model).

Step 1. (Initialization) Set $x_j = 0$, $j = 1, 2, \ldots, n$.

Set $s_j = v_j(1-d_j)$, $j = 1, 2, \ldots, n$.

Step 2. (Iterative step) Find k such that $s_k = \max_{1 \leq j \leq n} s_j$.

Reset $x_k = x_k + 1$.

Reset $M = M - 1$.

Step 3. (Termination test) If $M = 0$, stop. (Current solution is optimal.)

Otherwise, reset $s_k = d_k s_k$ and go to Step 2.

The second example derives from a promotion model due to Boza (1974). In an important subproblem in Boza's model, a given candidate population is partitioned into n distinct classification pools, with u_j candidates in the j^{th} pool, $j = 1, 2, \ldots, n$. In deciding how many candidates should be promoted from each classification, it is desired to minimize the squared error between the achieved classification proportions after promotion and certain predetermined target proportions. In addition, at least ℓ_j candidates must be promoted from the j^{th} pool, $j = 1, 2, \ldots, n$. When there is a total of V vacancies to be filled, the subproblem becomes the quadratic integer programming problem,[1/]

[1/] See McCallum (1974) for a more detailed description of the model.

$$\text{minimize} \quad \sum_{j=1}^{n} (a_j x_j^2 + b_j x_j + c_j) ,$$

subject to

$$\sum_{j=1}^{n} x_j = V \tag{8}$$

$$\ell_j \leq x_j \leq u_j, \quad x_j \quad \text{integer}, \quad j = 1, 2, \ldots , n .$$

While (8) is a minimization problem, it turns out that each $a_j > 0$, $j = 1, 2, \ldots , n$. Thus, (8) is a convex-separable integer program, and so can also be treated by the methods discussed in Section 2. The associated linear program (T) is again a bounded-variable knapsack problem:

$$\sum_{j=1}^{n} (a_j \ell_j^2 + b_j \ell_j + c_j) + \min \sum_{j=1}^{n} \sum_{k=1}^{u_j - \ell_j} s_{jk} x_{jk} ,$$

$$\text{subject to} \quad \sum_{j=1}^{n} \sum_{k=1}^{u_j - \ell_j} x_{jk} = V' \tag{9}$$

$$0 \leq x_{jk} \leq 1, \; k = 1,2,\ldots,u_j - \ell_j, \; j = 1,2,\ldots,n ,$$

where $V' = V - \sum_{j=1}^{n} \ell_j$. By (2),

$$s_{jk} = [2(\ell_j + k) - 1] a_j + b_j, \quad k = 1,2,\ldots,u_j - \ell_j, \quad j = 1,2,\ldots,n , \tag{10}$$

or, equivalently,

$$s_{jk+1} = s_{jk} + 2a_j, \qquad k = 1,2,\ldots,u_j - \ell_j, \quad j = 1,2,\ldots,n . \tag{11}$$

In the algorithm presented below, the coefficients s_{jk} are computed by (11). It is assumed that $\sum_{j=1}^{n} \ell_j < V < \sum_{j=1}^{n} u_j$, since otherwise the solution is trivial.

Algorithm 2 (Boza promotion model).

Step 1. (Initialization) Set $x_j = \ell_j$, $j = 1, 2, \ldots, n$.

Set $s_j = (2\ell_j+1)a_j + b_j$, $j = 1, 2, \ldots, n$.

Set $t_j = 2a_j$, $j = 1, 2, \ldots, n$.

Set $V' = V - \sum_{j=1}^{n} \ell_j$.

Step 2. (Iterative step) Find k such that $s_k = \min_{\{j \mid x_j < u_j\}} s_j$.

Reset $x_k = x_k + 1$.

Reset $V' = V' - 1$.

Step 3. (Termination test) If $V' = 0$, stop. (Current solution is optimal.)

Otherwise, reset $s_k = s_k + t_k$ and go to Step 3.

In determining the index k in Step 2, only those j for which $x_j < u_j$ are considered, since any variable x_j which is already at its upper bound obviously cannot be increased further.

Finally, the decomposition of block angular integer programs which have only a single linking constraint is a third application which gives rise to problems of the form (S_f). This problem and its solution procedure are discussed in detail by Kochman (1976), and so we shall not repeat it here. However, an interesting difference from the models of Boza and of Schwartz and Dym is that, in this case, each $f_j(x_j)$, $j = 1, 2, \ldots, n$, is naturally piecewise linear. Hence, it is not necessary to introduce the full set of $(u_j-\ell_j)$ zero-one variables x_{jk} $(k = 1,2,\ldots,u_j-\ell_j)$ in defining the approximating function $g_j(x_j)$.

Rather, auxiliary variables are added only at the non-integer breakpoints of $f_j(x_j)$.

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A class of nonlinear integer programs is introduced. Problems in this class are characterized by a concave and separable objective function subject to a set of linear constraints.

It is shown how by suitably modifying the objective function, the theory of spearability in linear programming can be applied to derive efficient solution procedures for problems falling in this class. This work unifies and extends

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20 Abstract

several results previously obtained independently in the literature. Two illustrative applications are discussed in some detail, and specified algorithms are presented for these examples.

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